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Induced representations of quantum kinematical algebras and quantum mechanics

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Abstract

Unitary representations of kinematical symmetry groups of quantum systems are fundamental in quantum theory. In this paper we propose their generalization to quantum kinematical groups. Using the method, proposed by us in a recent paper (Arratia O and del Olmo M A 2001 *Preprint* math.QA/0110275) to induce representations of quantum bicrossproduct algebras, we construct the representations of the family of standard quantum inhomogeneous algebras $U_\lambda(iso_\omega(2))$. This family contains the quantum Euclidean, Galilei and Poincaré algebras, all of them in (1 + 1) dimensions. As byproducts we obtain the actions of these quantum algebras on regular co-spaces that are an algebraic generalization of the homogeneous spaces and q-Casimir equations which play the role of q-Schrödinger equations.

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1. Introduction

The role played by the unitary representations of the (kinematical) symmetry groups of quantum physical systems is well known. One can classify the elementary systems [1] according to the unitary representations or obtain their corresponding Schrödinger equations using local representations [2]. In recent years we have been involved in a programme directed towards the construction of a theory of induced representations for quantum algebras [3–6]. Our aim is to obtain a quantum counterpart to the programme that Wigner started in 1939 for Lie groups [1], which has been so fruitful in quantum physics.

Kinematical groups such as Poincaré and Galilei, whose physical interest is not in any doubt, have the structure of a semidirect product. The quantum version of this kind of structure for quantum groups is that of the bicrossproduct [7, 8]. In this case a quantum Lie algebra

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inherits the 'semidirect' structure in the algebra sector and the algebra of the functions also has a semidirect product structure in the coalgebra sector.

It is worth mentioning that there exist several types of deformations of kinematical algebras, most of them sharing the bicrossproduct structure. In a first approximation we can consider standard and non-standard deformations, which are related to the fact that the associated classical r-matrix is quasi-triangular or triangular, respectively [9]. Inside the former we find 'time-like' deformations, in the sense that the generator of the time translations is primitive and the deformation parameter has dimensions of the inverse of time. This is the case for the κ-Poincaré [10] and κ-Galilei algebras [11], with κ denoting the deformation parameter. These κ-algebras have a bicrossproduct structure, which was pointed out in [12] for the κ-Poincaré algebra and in [13] for the κ-Galilei algebra.

There is now another standard 'space-like' deformation, since a space-translation generator remains primitive and the deformation parameter has dimensions of length, in contrast to the k-deformations. To perform these q-deformations the framework of the Cayley–Klein (CK) pseudo-orthogonal algebras [14] was used, which includes inhomogeneous algebras such as the Galilei, Poincaré and Euclidean algebras, in order to obtain a general and unified approach. The quantum deformation of these CK algebras in (1 + 1) dimensions was performed in [15]. The generalization to higher dimensions was presented in [16–18]. The deformation parameter of these q-algebras has dimensions of length. Moreover, these quantum CK inhomogeneous algebras have a bicrossproduct structure as studied in [19]. Incidentally, the κ-Poincaré algebra also appears inside the q-CK family after appropriately choosing the parameters and the 'physical' basis of the algebra [19].

The non-standard deformation of the Poincaré algebra in (1 + 1) dimensions appeared for the first time in [20], and the non-standard (1 + 1) Galilei–Heisenberg algebra was introduced in [21]. Both non-standard kinematical algebras also have a bicrossproduct structure, as displayed in [13]. Following with the above simile this non-standard deformation of the Poincaré algebra can be seen as a 'light-like' deformation if one considers a null-plane basis (see [22–24], respectively). All these quantum 'null-plane' Poincaré algebras share the bicrossproduct structure that was displayed in [13] for the (1 + 1) case and in [25] for higher dimensions. The non-standard Galilei algebras also have a bicrossproduct structure.

The interest of the quantum versions of the kinematical groups and algebras from a physical point of view is as 'quantum' generalizations of the symmetries of the physical space–time in a noncommutative framework. The study of these quantum symmetries and their representations generalizes the Wigner programme inside the perspective of the noncommutative geometry [26], whose importance in physics is increasing.

For these groups having the structure of a semidirect product, Mackey's method [27] provides their unitary representations by induction from the representations of one of their subgroups. In this paper we study the induced representations for some quantum kinematical algebras acting in spaces of (1 + 1)-dimensions using the CK approach, which allows us a unified point of view for all of them. The fact that these quantum CK inhomogeneous algebras have a bicrossproduct structure, which all of them share, simplifies the construction procedure of their induced representations and also gives a unified model for all of them.

The induction procedure formulated by us presents a strong algebraic character because we made use of objects such as modules, comodules, etc, which, from our point of view, are the appropriate tools to work with the algebraic structures characteristic of the quantum algebras and groups.

In the literature we can find some attempts to develop techniques that generalize the Mackey method [27] of induced representations for semidirect product groups to the quantum case. For instance, in [28] Dobrev presented a method for constructing representations of quantum groups near to ours, since both methods emphasize the dual case, closer to the classical one, and the representations are constructed in the algebra sector. Other authors [29–33] have also extended the induction technique to quantum groups but constructing corepresentations, i.e. representations of the coalgebra sector.

The organization of this paper is as follows. In section 2 we present a mathematical outline of the concepts that we will use throughout the paper in order to unify notation. Section 3 is devoted to summarizing the theory of induced representations of quantum bicrossproduct algebras. We begin to study the induction problem taking into account the deep relation between modules and representations obtaining, in some sense, deeper results from a geometric point of view using the concept of regular co-space. Since our aim is to construct the induced representations of certain quantum inhomogeneous CK algebras, section 4 is devoted to briefly describing these q-algebras. In section 5 we construct, in some detail, the induced representations following the method developed in section 3. We start with the computation of the flow associated with the action of the generator of one of the factors of the bicrossproduct over the other. After, we are able to determine the regular co-spaces determining the induced representations that we obtain below in an explicit way. Moreover, q-Casimir equations are also obtained.

2. Preliminaries

Let $H = (V; m\eta; \Delta\epsilon; S)$ be a Hopf algebra with underlying vector space V over the field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$), multiplication $m : H \otimes H \to H$, coproduct $\Delta : H \to H \otimes H$, unit $\eta : \mathbb{K} \to H$, counit $\epsilon : H \to \mathbb{K}$ and antipode $S : H \to H$.

The algebras involved in this work are infinite-dimensional algebras but finitely generated, for this reason we will use a multi-index notation [6]. Let A be an algebra generated by the elements $(a_1, a_2, \ldots, a_r)$ such that the ordered monomials $a^n = a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}$ ($n = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$) form a basis of the linear space underlying A. An arbitrary product of generators of A is written in a normal ordering if it is expressed in terms of the (ordered) basis $(a^n)_{n\in\mathbb{N}^r}$. The unit of A, 1_A, is denoted by a^0 ($0 \in \mathbb{N}^n$). Multi-factorials and multi-deltas are defined by

$$l! = \prod_{i=1}^{n} l_i!, \qquad \delta_l^m = \prod_{i=1}^{n} \delta_{l_i}^{m_i}. \tag{2.1}$$

A pairing between two Hopf algebras, H and H', is a bilinear mapping $\langle \cdot, \cdot \rangle : H \times H' \to \mathbb{K}$ verifying some defining relations [9]. The pairing is said to be left (right) nondegenerate if $[\langle h, \varphi\rangle = 0, \forall \varphi \in H'] \Rightarrow h = 0$ ($[\langle h, \varphi\rangle = 0, \forall h \in H] \Rightarrow \varphi = 0$). The pairing is said to be nondegenerate if it is simultaneously left and right nondegenerate. Two Hopf algebras and a nondegenerate pairing, $(H, H', \langle \cdot, \cdot \rangle)$, determine a 'nondegenerate triplet'. The bases (h^m) of H and (φ_n) of H' are dual with respect to the nondegenerate pairing if

$$\langle h^m, \varphi_n\rangle = c_n \delta_n^m, \qquad c_n \in \mathbb{K} - \{0\}. \tag{2.2}$$

Given a nondegenerate triplet and the map $f : H \to H$, we shall say that the map $f^\dagger : H' \to H'$, defined by

$$\langle h, f^\dagger(\varphi)\rangle = \langle f(h), \varphi\rangle, \tag{2.3}$$

is the adjoint map to f with respect to $\langle \cdot, \cdot \rangle$.

We will use the endomorphisms of 'multiplication' (denoted again by h and φ) and 'formal derivation' ($\frac{\partial}{\partial h} \equiv \partial_h$ and $\frac{\partial}{\partial \varphi} \equiv \partial_\varphi$). They verify that $h^\dagger = \partial_\varphi$ and $\varphi^\dagger = \partial_h$. The situation is similar when H and H' are finitely generated. The definition of the formal derivative is

$$\frac{\partial}{\partial h_i}(h_1^{l_1}\cdots h_i^{l_i}\cdots h_n^{l_n}) = l_i h_1^{l_1}\cdots h_i^{l_i-1}\cdots h_n^{l_n}. \tag{2.4}$$

The generalization of the 'multiplication' operators is not straightforward if the algebras are non-commutative. To avoid any confusion the formal operators associated with the generators h_i will be denoted using a bar over the corresponding symbol. The action of these operators is

$$\overline{h}_i(h_1^{l_1}\cdots h_i^{l_i}\cdots h_n^{l_n}) = h_1^{l_1}\cdots h_i^{l_i+1}\cdots h_n^{l_n}. \tag{2.5}$$

For the elements $\varphi^i \in H'$ the 'multiplication' and derivative operators are defined in a similar way. Note that if H (resp. H') is commutative then $\overline{h}_i$ (resp. $\overline{\varphi}^i$) acts as a multiplication operator, but this is no longer true when the algebra is noncommutative. The adjoint operators are $\overline{h}_i^\dagger = \partial_{\varphi^i}$, $\overline{\varphi}^{i\dagger} = \partial_{h_i}$. The commutation relations for $\overline{h}_i$ and ∂_{h_i} (similar for $\overline{\varphi}^i$ and ∂_{φ^i}) are

$$[\partial_{h_i}, \overline{h}_j] = \delta_{ij}, \qquad [\overline{h}_i, \overline{h}_j] = 0, \qquad [\partial_{h_i}, \partial_{h_j}] = 0. \tag{2.6}$$

Let (V, α, A) be a triad composed of a unital and associative $\mathbb{K}$-algebra A, a $\mathbb{K}$-vector space V and a linear map (called action) $\alpha : A \otimes_{\mathbb{K}} V \to V$ ($\alpha(a \otimes v) = a \triangleright v$). It is said that (V, α, A) (or $(V, \triangleright, A)$) is a left A-module if

$$a \triangleright (b \triangleright v) = (ab) \triangleright v, \qquad 1 \triangleright v = v, \qquad \forall a, b \in A, \forall v \in V. \tag{2.7}$$

Dualizing an A-module a comodule is obtained. The triad $(V, \blacktriangleleft, C)$, where C is an associative $\mathbb{K}$-coalgebra with counit, V a $\mathbb{K}$-vector space and $\blacktriangleleft$: $V \to C \otimes_{\mathbb{K}} V$ ($v \blacktriangleleft = v^{(1)} \otimes v^{(2)}$) a (linear map) coaction, is said to be a left C-comodule if

$$v^{(1)}{}_{(1)} \otimes v^{(1)}{}_{(2)} \otimes v^{(2)} = v^{(1)} \otimes v^{(2)}{}_{(1)} \otimes v^{(2)}{}_{(2)}, \qquad \epsilon(v^{(1)})v^{(2)} = v, \qquad \forall v \in V, \tag{2.8}$$

with $\Delta(c) = c_{(1)} \otimes c_{(2)}$ denoting the coproduct of the elements of C.

A morphism of the left A-modules, $(V, \triangleright, A)$ and $(V', \triangleright', A)$, is a linear map, $f : V \to V'$, equivariant with respect the action, i.e.

$$f(a \triangleright v) = a \triangleright' f(v), \qquad \forall a \in A, \forall v \in V. \tag{2.9}$$

A linear map $f : V \to V'$ between two C-comodules, $(V, \blacktriangleleft, C)$ and $(V', \blacktriangleleft', C)$ is a morphism if

$$v^{(1)} \otimes f(v^{(2)}) = f(v)^{(1)'} \otimes f(v)^{(2)'}, \qquad \forall v \in V. \tag{2.10}$$

If a bialgebra acts or coacts on a vector space equipped with an additional structure (algebra, coalgebra or bialgebra) some compatibility relations for the action may be required [8].

Let A, B, C be an algebra, a bialgebra and a coalgebra, respectively. The left module $(A, \triangleright, B)$ is a B-module algebra if m_A and η_A are morphisms of B-modules. That is, if

$$b \triangleright (aa') = (b_{(1)} \triangleright a)(b_{(2)} \triangleright a'), \qquad b \triangleright 1 = \epsilon(b)1, \qquad \forall b \in B, \forall a, a' \in A. \tag{2.11}$$

A left B-module $(C, \triangleright, B)$ is a B-module coalgebra if Δ_C and ϵ_C are morphisms of B-modules, i.e. if

$$(b \triangleright c)_{(1)} \otimes (b \triangleright c)_{(2)} = (b_{(1)} \triangleright c_{(1)}) \otimes (b_{(2)} \triangleright c_{(2)}), \qquad \epsilon_C(b \triangleright c) = \epsilon_B(b)\epsilon_C(c),$$
$$\forall b, c \in B.$$

A left B-comodule $(C, \blacktriangleleft, B)$ is said to be a B-comodule coalgebra if Δ_C and ϵ_C are morphisms of B-comodules, i.e.

$$c^{(1)} \otimes c^{(2)}{}_{(1)} \otimes c^{(2)}{}_{(2)} = c_{(1)}{}^{(1)} c_{(2)}{}^{(1)} \otimes c_{(1)}{}^{(2)} \otimes c_{(2)}{}^{(2)}, \qquad c^{(1)}\epsilon_C(c^{(2)}) = (\eta_B \circ \epsilon_C)(c).$$

A left B-comodule $(A, \blacktriangleleft, B)$ is a B-comodule algebra if m_A and η_A are morphisms of B-comodules. Explicitly

$$(aa')_{(1)} \otimes (aa')_{(2)} = a_{(1)}a'_{(1)} \otimes a_{(2)}a'_{(2)}, \qquad 1_A \blacktriangleleft = 1_B \otimes 1_A. \tag{2.12}$$

The triad $(B', \triangleright, B)$ is a left B-module bialgebra if simultaneously it is a B-module algebra and a B-module coalgebra; $(B', \blacktriangleleft, B)$ is a left B-comodule bialgebra if it is a B-comodule algebra and a B-comodule coalgebra.

A regular module (comodule) is an A-module (C-comodule) whose vector space is the underlying vector space of the algebra A (coalgebra C). The action (coaction) is defined in terms of the algebra product (coalgebra coproduct). If B is a bialgebra, the regular B-module $(B, \triangleright, B)$, whose regular action is $b \triangleright b' = bb'$, is a module coalgebra. The regular module $(B^*, \triangleleft, B)$, obtained by dualization, is a module algebra with regular action $\varphi \triangleleft b = \langle \varphi_{(1)}, b\rangle \varphi_{(2)}$ with $b \in B, \varphi \in B^*$.

Let K and L be two Hopf algebras, with $(L, \triangleleft, K)$ a right K-module algebra and $(K, \blacktriangleleft, L)$ a left L-comodule coalgebra. The tensor product $K \otimes L$ is equipped simultaneously with the semidirect structures of the algebra $K \ltimes L$ and coalgebra $K \rtimes\!\!\blacktriangleleft L$. If some compatibility conditions are verified $K \ltimes L$ and $K \rtimes\!\!\blacktriangleleft L$ determine a Hopf algebra called the (right–left) bicrossproduct and denoted by $K \triangleright\!\!\blacktriangleleft L$ [7,8].

Let $\mathcal{K}$ and $\mathcal{L}$ be two Hopf algebras and $(\mathcal{L}, \triangleright, \mathcal{K})$ and $(\mathcal{K}, \blacktriangleright, \mathcal{L})$ a left K-module algebra and a right L-comodule coalgebra, respectively, verifying the corresponding compatibility conditions. Then $\mathcal{L} \rtimes \mathcal{K}$ and $\mathcal{L} \blacktriangleright\!\!< \mathcal{K}$ determine a Hopf algebra, $\mathcal{L} \blacktriangleright\!\!\triangleleft \mathcal{K}$, called the (left–right) bicrossproduct.

These two bicrossproduct structures are related by duality: if K and L are finite-dimensional bialgebras and the K-module algebra $(L, \triangleleft, K)$ and the L-comodule coalgebra $(K, \blacktriangleleft, L)$ verify the compatibility conditions determining the bicrossproduct $K \triangleright\!\!\blacktriangleleft L$, then $(K \triangleright\!\!\blacktriangleleft L)^* = K^* \blacktriangleright\!\!\triangleleft L^*$.

It has been proved in [3] that dual bases and $*$-structures over bicrossproduct Hopf algebras may be constructed when the corresponding bases and $*$-structures of the bicrossproduct factors are known. Thus, let $H = K \triangleright\!\!\blacktriangleleft L$ be a bicrossproduct Hopf algebra and $(K, K^*, \langle\cdot,\cdot\rangle_1)$ and $(L, L^*, \langle\cdot,\cdot\rangle_2)$ be nondegenerate triplets, then the expression

$$\langle kl, \kappa\lambda\rangle = \langle k, \kappa\rangle_1 \langle l, \lambda\rangle_2 \tag{2.13}$$

defines a nondegenerate pairing between H and H^*. If (k_m) and (κ_m) are dual bases for K and K^*, and (l_n) and (λ_n) for L and L^*, then $(k_m l_n)$ and $(\kappa^m \lambda^n)$ are dual bases for H and H^*.

It can be proved that given a bicrossproduct Hopf algebra, $H = K \triangleright\!\!\blacktriangleleft L$, such that K and L are equipped with $*$-structures verifying the compatibility relation $(l \triangleleft k)^* = l^* \triangleleft S(k)^*$, there is a $*$-structure on the algebra sector of H determined by

$$(kl)^* = l^* k^*, \qquad k \in K, l \in L. \tag{2.14}$$

3. Induced representations of bicrossproduct algebras

The main results regarding the theory of induced representations for quantum bicrossproduct algebras will be summarized in this section (see [3–6] for a complete description of induced representations of quantum algebras).

As we have seen in the previous section, different actions are involved, and henceforth we will denote them by the following symbols (or their symmetric for the corresponding right actions and coactions): $\triangleright$ ($\blacktriangleleft$) (bicrossproduct actions (coactions)), $\vdash$ (induced and inducting representations) and $\succ$, $\prec$ (regular actions).

Let $(H, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a nondegenerate triplet and L a commutative subalgebra of H. Suppose that $\{l_1, \ldots, l_s\}$ is a system of generators of L which is completed with $\{k_1, \ldots, k_r\}$ to get a system of generators of H, such that $(l_n)_{n\in\mathbb{N}^s}$ is a basis of L and $(k_m l_n)_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s}$ a basis of H. Moreover, there is a generator system of $\mathcal{H}$, $\{\kappa_1, \ldots, \kappa_r, \lambda_1, \ldots, \lambda_s\}$, such that $(\kappa^m\lambda^n)_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s}$ is a basis of $\mathcal{H}$ dual to that of H with pairing

$$\langle k_m l_n, \kappa^{m'}\lambda^{n'}\rangle = m!n!\delta_m^{m'}\delta_n^{n'}. \tag{3.1}$$

In order to construct the representation of H induced by the character of L, determined by $a = (a_1, \ldots, a_s) \in \mathbb{K}^s$ (explicitly, $1 \dashv l_n = a_n = a_1^{n_1}\cdots a_s^{n_s}, n \in \mathbb{N}^s$), we need to know its carrier space $\mathbb{K}^\uparrow$ and the action of H on it. The elements of $\mathbb{K}^\uparrow$ are those of $\mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ verifying the invariance condition

$$f(hl) = f(h) \dashv l, \qquad \forall l \in L, \forall h \in H. \tag{3.2}$$

They can be written as

$$f = \sum_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s} f_{mn}\kappa^m\lambda^n \tag{3.3}$$

by identifying $\mathbb{K}^\uparrow = \mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ with $\mathcal{H}$ via the pairing. The equivariance condition (3.2)

$$\langle hl, f\rangle = \langle h, f\rangle \dashv l, \qquad \forall l \in L, \forall h \in H, \tag{3.4}$$

together with duality give the relation

$$m!n!f_{mn} = \langle k_m l_n, f\rangle = \langle k_m, f\rangle a_n = m!f_{m0}a_n. \tag{3.5}$$

Hence, the elements of $\mathbb{K}^\uparrow$ are

$$f = \kappa\psi, \qquad \kappa \in \mathcal{K}, \psi = \mathrm{e}^{a_1\lambda_1}\cdots\mathrm{e}^{a_s\lambda_s}, \tag{3.6}$$

where $\mathcal{K}$ is the subspace of $\mathcal{H}$ generated by the linear combinations of the ordered monomials $(\kappa^m)_{m\in\mathbb{N}^r}$. Because ψ is product of exponentials, $\mathcal{K}$ and $\mathbb{K}^\uparrow$ are isomorphic ($\kappa \to \kappa\psi$).

The action of H on $\mathbb{K}^\uparrow$ is determined knowing the action over the basis elements $(\kappa^p\psi)_{p\in\mathbb{N}^r}$. Putting

$$(\kappa^p\psi) \dashv h = \sum_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s} [h]^p_{mn}\kappa^m\lambda^n, \qquad p \in \mathbb{N}^r, \tag{3.7}$$

the $[h]^p_{mn}$ coefficients are evaluated by means of the duality

$$m!n![h]^p_{mn} = \langle(\kappa^p\psi) \dashv h, k_m l_n\rangle = \langle\kappa^p\psi, hk_m l_n\rangle = \langle\kappa^p\psi, hk_m\rangle a_n. \tag{3.8}$$

The properties of the action allow one to compute it only for the generators of H instead of considering an arbitrary element of H. The problem reduces to writing hk_m in normal ordering to get the value of the paring in (3.8). Since in many cases this task is very cumbersome, our objective will be to take advantage of the bicrossproduct structure to simplify the computations.

In the following we will restrict ourselves to Hopf algebras having a bicrossproduct structure such as $H = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$, such that $\mathcal{K}$ is cocommutative and $\mathcal{L}$ commutative. Let us suppose that the algebras $\mathcal{K}$ and $\mathcal{L}$ are finite generated by the sets $\{k_i\}_{i=1}^r$ and $\{l_i\}_{i=1}^s$, respectively, the k_i are primitive and $(k_n)_{n\in\mathbb{N}^r}$ and $(l_m)_{m\in\mathbb{N}^s}$ are bases of the vector spaces underlying $\mathcal{K}$ and $\mathcal{L}$, respectively. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be the dual algebras of $\mathcal{K}$ and $\mathcal{L}$ having dual systems to those of $\mathcal{K}$ and $\mathcal{L}$ with analogous properties to them. So, duality between H and H^* is given by (3.1).

We are interested in the construction of the representations induced by 'real' characters of the commutative sector $\mathcal{L}$. We will show that the solution of this problem can be reduced to the study of certain dynamical systems which present, in general, a nonlinear action. The above results can be summarized in the following theorem.

Theorem 3.1. *The carrier space of the representation of H induced by the character a of $\mathcal{L}$,*

$$1 \dashv l_n = a_n, \qquad a \in \mathbb{C}^s, n \in \mathbb{N}^s, \tag{3.9}$$

is isomorphic to $\mathcal{K}^*$ and is constituted by the elements of the form

$$\kappa\psi, \qquad \kappa \in \mathcal{K}^*, \qquad \psi = \mathrm{e}^{a_1\lambda_1}\mathrm{e}^{a_2\lambda_2}\cdots\mathrm{e}^{a_s\lambda_s}, \qquad \lambda_i \in \mathcal{L}^*. \tag{3.10}$$

The induced action of the elements of H over the elements of the carrier space $\mathbb{C}^\uparrow$ is given by

$$f \dashv h = \sum_{m\in\mathbb{N}^r} \kappa^m \left\langle h\frac{k_m}{m!}, f\right\rangle \psi, \qquad f \in \mathbb{C}^\uparrow. \tag{3.11}$$

To obtain the explicit action of the generators of $\mathcal{K}$ and $\mathcal{L}$ in the induced representation, let us start by identifying $\mathcal{L}$, because it is commutative, with the algebra of functions $F(\mathbb{R}^s)$ by the morphism $\mathcal{L} \to F(\mathbb{R}^s)$, $(l \mapsto \tilde{l})$, mapping the generators of $\mathcal{L}$ into the canonical projections

$$\tilde{l}_j(x) = x_j, \qquad \forall x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s, \ j = 1, 2, \ldots, s. \tag{3.12}$$

This identification allows one to choose a $*$-structure keeping invariant the generators of $\mathcal{L}$ by

$$l_j^* = l_j, \qquad j = 1, 2, \ldots, s. \tag{3.13}$$

Hence, the characters of $\mathcal{L}$ (3.9) compatible with (3.13) are real. They can now be written as

$$1 \dashv l = \tilde{l}(a), \qquad a \in \mathbb{R}^s. \tag{3.14}$$

Also, the right action of $\mathcal{K}$ on $\mathcal{L}$ can be carried to $F(\mathbb{R}^s)$. Since the generators of $\mathcal{K}$ are primitive, they act by derivations on the $\mathcal{K}$-module algebra of $\mathcal{K} \rhd\!\!\blacktriangleleft \mathcal{L}$ inducing vector fields, X_i, on $\mathbb{R}^s$ by

$$X_i\tilde{l} = \widetilde{l \lhd k_i}, \qquad i = 1, 2, \ldots, r. \tag{3.15}$$

The flow associated with X_i, $\Phi_i : \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^s$, is given by

$$(X_i f)(x) = (Df_{x,\Phi_i})(0), \tag{3.16}$$

where $f_{x,\Phi_i}(t) = f \circ \Phi_i^t(x)$ and D is the derivative operator. Thus, we can state the following theorem that allows us to have explicit formulae for the action.

Theorem 3.2. *The explicit action of the generators of $\mathcal{K}$ and $\mathcal{L}$ in the induced representation determined in theorem 3.1 and realized in the space $\mathcal{K}^*$ is given by the following expressions:*

$$\begin{aligned} \kappa \dashv k_i &= \kappa \prec k_i, \\ \kappa \dashv l_j &= \kappa\hat{l}_j \circ \Phi_{(\kappa_1,\kappa_2,\ldots,\kappa_r)}(a), \end{aligned} \tag{3.17}$$

where $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$, the symbol $\prec$ denotes the regular action of $\mathcal{K}$ on $\mathcal{K}^*$ and $\Phi_{(\kappa_1,\kappa_2,\ldots,\kappa_r)} = \Phi_r^{\kappa_r} \circ \cdots \circ \Phi_2^{\kappa_2} \circ \Phi_1^{\kappa_1}$.

We can reformulate the induction procedure in terms of modules because of their deep relation with representations [3,6]. The regular H-modules associated with a Hopf algebra H and its dual H^*, $(H, \prec, H)$, $(H^*, \succ, H)$, $(H, \succ, H)$ and $(H^*, \prec, H)$, will help us to describe the actions as well as the carrier spaces involved in the induced representations. The modules $(H^*, \succ, H)$ and $(H^*, \prec, H)$ are called co-spaces since they can be considered as an algebraic generalization of the concept of G-space.

The commutative or cocommutative Hopf algebras are, in fact, of the form $F(G)$ or $\mathbb{C}[G]$, the group algebra of G, (or $U(\mathfrak{g})$) for any group G [8]. Hence, the bicrossproduct Hopf algebras, that we consider in this paper, can be factorized as

$$H = U(\mathfrak{k}) \rhd\!\!\blacktriangleleft F(L), \qquad H^* = F(K) \blacktriangleright\!\!\lhd U(\mathfrak{l}) \tag{3.18}$$

where K and L are Lie groups with associated Lie algebras $\mathfrak{k}$ and $\mathfrak{l}$, respectively.

The use of elements of H and H^*, such as

$$\begin{aligned} &k\lambda \in H, &&k \in K, \lambda \in F(L),\\ &\kappa l \in H^*, &&\kappa \in F(K), l \in L, \end{aligned} \tag{3.19}$$

instead of the standard bases of ordered monomials, allows an effective description of the H regular modules.

Theorem 3.3. *The action on each of the four regular H-modules is*

$$\begin{aligned} &(H, \prec, H) : (k\lambda) \prec k' = kk'(\lambda \triangleleft k'), && (k\lambda) \prec \lambda' = k\lambda\lambda';\\ &(H^*, \succ, H) : k' \succ (\kappa l) = (k' \succ \kappa)(k' \triangleright l), && \lambda' \succ (\kappa l) = \lambda'(l)\kappa l;\\ &(H, \succ, H) : k' \succ (k\lambda) = k'k\lambda, && \lambda' \succ (k\lambda) = k(\lambda' \triangleleft k)\lambda;\\ &(H^*, \prec, H) : (\kappa l) \prec k' = (\kappa \prec k')l, && (\kappa l) \prec \lambda' = \kappa(\lambda' \circ \hat{l})l; \end{aligned} \tag{3.20}$$

where $\hat{l}$ is the map, $K \to L(k \mapsto k \triangleright l)$, projecting the group K on the orbit passing through $l \in L$ such that $\langle l^{(1)}, \lambda\rangle l^{(2)} = \lambda \circ \hat{l}$, for all $k, k' \in K$, $\lambda, \lambda' \in F(L)$, $\kappa \in F(K)$ and $l \in L$.

Note that in this theorem we have taken into account that in $(U(\mathfrak{l}), \succ, F(L))$ holds

$$\lambda \succ l = \lambda(l)l, \qquad \forall\lambda \in F(L), \forall l \in L. \tag{3.21}$$

Now we can carry out a complete analysis of the representations of $H = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(L)$ induced by the one-dimensional modules of its commutative sector. Note that the set of characters of the algebra $F(L)$ is its spectrum and, hence, the spectrum of $F(L)$ is isomorphic to L. Fixing an element l of L, the character is given by

$$1 \dashv \lambda = \lambda(l), \qquad \lambda \in F(L). \tag{3.22}$$

The carrier space $\mathbb{C}^{\uparrow}$ of the representation of $H = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(L)$ induced by (3.22) is the set of elements $f \in H^*$ satisfying the condition

$$\lambda \succ f = \lambda(l)f, \qquad \forall\lambda \in F(L). \tag{3.23}$$

Expanding f in terms of the bases of $\mathfrak{k}$ and $\mathfrak{l}$, imposing the equivariance condition and considering the definition of second kind coordinates λ_j over the group L we obtain that

$$f = \kappa l, \qquad \kappa \in F(K). \tag{3.24}$$

The right regular action determines the action on the induced module that can be carried to $F(K)$ using the isomorphism $F(K) \to \mathbb{C}^{\uparrow}(\kappa \mapsto \kappa l)$. Consequently,

$$\kappa \dashv k = \kappa \prec k, \qquad \kappa \dashv \lambda = \kappa(\lambda \circ \hat{l}). \tag{3.25}$$

Comparing these expressions with those of theorem 3.2, we see that the action of $U(\mathfrak{k})$ is determined by the regular action. The action of $F(L)$ is multiplicative and its evaluation is essentially reduced to obtain the flows associated with the action of K on L derived from the bicrossproduct structure of H.

Finally, the following theorem summarizes the induction procedure for bicrossproduct algebras.

Theorem 3.4. *Let us consider an element $l \in L$ supporting a global action of the group K. The carrier space, $\mathbb{C}^{\uparrow}$, of the representation of H induced by the character determined by l is the set of elements of H^* of the form*

$$\kappa l, \qquad \kappa \in F(K). \tag{3.26}$$

There is an isomorphism between $\mathbb{C}^{\uparrow}$ and $F(K)$ given by the map $\kappa \mapsto \kappa l$. The action induced by the elements of the form $k \in K$ and $\lambda \in F(L)$ in the space $F(K)$ is

$$\begin{aligned} \kappa \dashv k &= \kappa \prec k \\ \kappa \dashv \lambda &= \kappa(\lambda \circ \hat{l}). \end{aligned} \tag{3.27}$$

The modules induced by l and $k \triangleright l$ are isomorphic. Consequently, the induction algorithm establishes a correspondence between the space of orbits L/K and the set of equivalence classes of representations.

3.1. Local representations

The quantum counterpart of the local representations [2] of Lie groups can be obtained inducing from representations of the subalgebra $U(\mathfrak{k})$. Given a character κ of $U(\mathfrak{k})$,

$$\kappa \in \text{spectrum}U(\mathfrak{k}) \subset F(K), \qquad k \vdash 1 = \kappa(k), \tag{3.28}$$

the carrier space, $\mathbb{C}^{\uparrow}$, of the representation induced by κ is determined by the equivariance condition

$$f \prec k = \kappa(k) f, \qquad \forall k \in U(\mathfrak{k}) \tag{3.29}$$

obtaining that the elements of $\mathbb{C}^{\uparrow}$ are of the form

$$f = \kappa l, \qquad l \in U(\mathfrak{l}). \tag{3.30}$$

The isomorphism $U(\mathfrak{l}) \to \mathbb{C}^{\uparrow}$ $(l \mapsto \kappa l)$ allows one to realize the induced representation over $U(\mathfrak{l})$

$$k \vdash l = \kappa(k) k \triangleright l, \qquad \lambda \vdash l = \lambda(l) l. \tag{3.31}$$

The local representations of the quantum extended (1 + 1) Galilei algebra [33] have been obtained in [34].

4. Quantum $\mathfrak{iso}_\omega(2)$ algebras

The algebras called CK pseudo-orthogonal algebras $\mathfrak{so}_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ are a family of $(N+1)N/2$ dimensional real Lie algebras characterized by N real parameters $(\omega_1, \omega_2, \ldots, \omega_N)$ [14, 19]. In the 'geometric' basis $(J_{ij})_{0 \leqslant i < j \leqslant N}$ the nonvanishing commutators are

$$[J_{ij}, J_{ik}] = \omega_{ij} J_{jk}, \qquad [J_{ij}, J_{jk}] = -J_{ik}, \qquad [J_{ik}, J_{jk}] = \omega_{jk} J_{ij}, \tag{4.1}$$

with $0 < i < j < k < N$ and $\omega_{ij} = \prod_{s=i+1}^{j} \omega_s$.

The parameters ω_i, in fact, only take the values 1, 0 and -1 since the generators J_{ij} can be rescaled. When $\omega_i \neq 0 \forall i$, the Lie algebra $\mathfrak{so}_{\omega_1,\omega_2,\ldots,\omega_N}(N+1)$ is isomorphic to some of the pseudo-orthogonal algebras $\mathfrak{so}(p,q)$ with $p+q = N+1$ and $p \geqslant q \geqslant 0$. If some ω_i vanish the corresponding algebra is inhomogeneous.

In this paper we are interested in the particular case of $N = 2$ and $\omega_1 = 0$. These inhomogeneous algebras $\mathfrak{so}_{0,\omega_2}(3)$ can be realized as algebras of groups of affine transformations on $\mathbb{R}^2$ [14] (i.e. the Euclidean group when $\omega_2 = 1$, the Galilei group when $\omega_2 = 0$ and the Poincaré group when $\omega_2 = -1$). In this case the generators J_{0i} are denoted by P_i stressing their role as translation generators. The remaining generator $J_{12} \equiv J$ is associated with compact ($\omega_2 = 1$) and noncompact rotations (Galilean and Lorentzian boosts).

The algebra $\mathfrak{so}_{0,\omega_2}(3)$, that we shall denote by $\mathfrak{iso}_\omega(2)$, is characterized by the commutators

$$[J, P_1] = P_2, \qquad [J, P_2] = -\omega P_1, \qquad [P_1, P_2] = 0. \tag{4.2}$$

We shall use the following 'generalized trigonometric functions' [14, 15]:

$$\mathrm{C}_\omega(x) = \frac{\mathrm{e}^{\sqrt{-\omega}x} + \mathrm{e}^{-\sqrt{-\omega}x}}{2}, \qquad \mathrm{S}_\omega(x) = \frac{\mathrm{e}^{\sqrt{-\omega}x} - \mathrm{e}^{-\sqrt{-\omega}x}}{2\sqrt{-\omega}}. \tag{4.3}$$

When $\omega < 0$ ($\omega > 0$) these expressions become the trigonometric (hyperbolic) functions. For $\omega = 0$ the parabolic functions $\mathrm{C}_0(x) = 1$ and $\mathrm{S}_0(x) = x$ are obtained. The functions (4.3) satisfy identities similar to those of the usual trigonometric functions. Some useful properties that will be used in following computations are

$$\begin{aligned}
&\mathrm{C}^2_\omega(x) + \omega\mathrm{S}^2_\omega(x) = 1,\\
&\mathrm{C}_\omega(x+y) = \mathrm{C}_\omega(x)\mathrm{C}_\omega(y) - \omega\mathrm{S}_\omega(x)\mathrm{S}_\omega(y),\\
&\mathrm{S}_\omega(x+y) = \mathrm{S}_\omega(x)\mathrm{C}_\omega(y) + \mathrm{C}_\omega(x)\mathrm{S}_\omega(y),\\
&C'_\omega(x) = -\omega\mathrm{S}_\omega(x), \qquad S'_\omega(x) = \mathrm{C}_\omega(x).
\end{aligned} \tag{4.4}$$

A simultaneous standard deformation for all the enveloping algebras $U(\mathfrak{so}_{\omega_1,\omega_2}(3))$ was introduced in [15], The particular case of the deformed Hopf algebras $U_\lambda(\mathfrak{iso}_\omega(2))$ is obtained taking $\omega_1 = 0$ and, obviously, $\omega_2 = \omega$.

It was proved in [19] that the standard quantum Hopf algebras $U_\lambda(\mathfrak{iso}_{\omega_2,\omega_3,\ldots,\omega_N}(N))$ have a bicrossproduct structure. So, $U_\lambda(\mathfrak{iso}_\omega(2))$ is characterized in a basis adapted to its bicrossproduct structure by

$$\begin{aligned}
&[J, P_1] = \frac{1-\mathrm{e}^{-2\lambda P_2}}{2\lambda} + \frac{1}{2}\lambda\omega P_1^2, \qquad [J, P_2] = -\omega P_1;\\
&\Delta(P_1) = P_1 \otimes 1 + \mathrm{e}^{-\lambda P_2} \otimes P_1,\\
&\Delta(P_2) = P_2 \otimes 1 + 1 \otimes P_2, \qquad \Delta(J) = J \otimes 1 + \mathrm{e}^{-\lambda P_2} \otimes J;\\
&\epsilon(P_1) = \epsilon(P_2) = \epsilon(J) = 0;\\
&S(P_1) = -\mathrm{e}^{\lambda P_2}P_1, \qquad S(P_2) = -P_2, \qquad S(J) = -\mathrm{e}^{\lambda P_2}J.
\end{aligned}$$

The bicrossproduct structure $U_\lambda(\mathfrak{iso}_\omega(2)) = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$, where $\mathcal{L}$ is the Hopf subalgebra spanned by (P_1, P_2) and $\mathcal{K}$ is the commutative and cocommutative Hopf algebra generated by J, is determined by the action of $\mathcal{K}$ on $\mathcal{L}$ given by

$$P_1 \triangleleft J = [P_1, J] = -\left[\frac{1-\mathrm{e}^{-2\lambda P_2}}{2\lambda} + \frac{1}{2}\lambda\omega P_1^2\right], \qquad P_2 \triangleleft J = [P_2, J] = \omega P_1, \tag{4.5}$$

and the left coaction of $\mathcal{L}$ on $\mathcal{K}$, which over the generator of $\mathcal{K}$ takes the value

$$J \blacktriangleleft = \mathrm{e}^{-\lambda P_2} \otimes J. \tag{4.6}$$

The dual algebra $F_\lambda(ISO_\omega(2))$ is generated by the local coordinates φ, a_1, a_2. Its commutators, coproduct, counit and antipode are given by

$$\begin{aligned}
&[a_1, \varphi] = \lambda(1 - \mathrm{C}_\omega(\varphi)), \qquad [a_2, \varphi] = \lambda\mathrm{S}_\omega(\varphi), \qquad [a_1, a_2] = \lambda a_1;\\
&\Delta(a_1) = a_1 \otimes \mathrm{C}_\omega(\varphi) + a_2 \otimes \omega\mathrm{S}_\omega(\varphi) + 1 \otimes a_1,\\
&\Delta(a_2) = -a_1 \otimes \mathrm{S}_\omega(\varphi) + a_2 \otimes \mathrm{C}_\omega(\varphi) + 1 \otimes a_2,\\
&\Delta(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi;\\
&\epsilon(a_1) = \epsilon(a_2) = \epsilon(\varphi) = 0;\\
&S(a_1) = -\mathrm{C}_\omega(\varphi)a_1 - \omega\mathrm{S}_\omega(\varphi)a_2, \qquad S(a_2) = \mathrm{S}_\omega(\varphi)a_1 - \mathrm{C}_\omega(\varphi)a_2, \qquad S(\varphi) = -\varphi.
\end{aligned} \tag{4.7}$$

This Hopf algebra exhibits the bicrossproduct structure $F_\lambda(ISO_\omega(2)) = \mathcal{K}^* \blacktriangleright\!\!\triangleleft \mathcal{L}^*$, dual of the one above, with $\mathcal{K}^*$ generated by φ and $\mathcal{L}^*$ by a_1, a_2. The left action of $\mathcal{L}^*$ over $\mathcal{K}^*$ is given by

$$a_1 \triangleright \varphi = \lambda(1 - \mathrm{C}_\omega(\varphi)), \qquad a_2 \triangleright \varphi = \lambda\mathrm{S}_\omega(\varphi). \tag{4.8}$$

The right coaction of $\mathcal{K}^*$ over $\mathcal{L}^*$ takes the following values over the generators of $\mathcal{L}^*$:

$$\blacktriangleright a_1 = a_1 \otimes \mathrm{C}_\omega(\varphi) + a_2 \otimes \omega \mathrm{S}_\omega(\varphi), \qquad \blacktriangleright a_2 = -a_1 \otimes \mathrm{S}_\omega(\varphi) + a_2 \otimes \mathrm{C}_\omega(\varphi). \tag{4.9}$$

The results mentioned in the last paragraph of section 2 dealing with dual bases and $*$-structures over bicrossproduct Hopf algebras allow one to construct a pair of dual bases in such a way that the duality form between $U_\lambda(\mathfrak{iso}_\omega(2))$ and $F_\lambda(ISO_\omega(2))$ is

$$\langle J^m P_1^n P_2^p, \varphi^q a_1^r a_2^s \rangle = m!n!p!\delta_m^q \delta_n^r \delta_p^s. \tag{4.10}$$

5. Representations of $U_\lambda(\mathfrak{iso}_\omega(2))$

Firstly, according to the theory of representations of bicrossproduct algebras displayed in section 3 and, in particular, theorem 3.2, we need to know the flow associated with the action of $\mathcal{K}$ on $\mathcal{L}$.

5.1. *One-parameter flow for* $U_\lambda(\mathfrak{iso}_\omega(2))$

In [19] the factor $\mathcal{K}$ is interpreted as the enveloping algebra $U(\mathfrak{so}_\omega(2))$, while $\mathcal{L}$ is seen as a deformation of the algebra of the group of the translations in the plane T_2, $U_\lambda(\mathfrak{t}_2)$. Hence,

$$U_\lambda(\mathfrak{iso}_\omega(2)) = U(\mathfrak{so}_\omega(2)) \triangleright\!\!\blacktriangleleft U_\lambda(\mathfrak{t}_2), \qquad F_\lambda(ISO_\omega(2)) = F(SO_\omega(2)) \blacktriangleright\!\!\triangleleft F_\lambda(T_2). \tag{5.1}$$

However, our interpretation given here is different since

$$U_\lambda(\mathfrak{iso}_\omega(2)) = U(\mathfrak{so}_\omega(2)) \triangleright\!\!\blacktriangleleft F(T_{\lambda,2}), \tag{5.2}$$

where $T_{\lambda,2}$ denotes the Lie group with composition law

$$(\alpha_1', \alpha_2')(\alpha_1, \alpha_2) = (\alpha_1' + \mathrm{e}^{-\lambda\alpha_2'}\alpha_1, \alpha_2' + \alpha_2). \tag{5.3}$$

Now the generators P_1 and P_2 are considered as a global coordinate system over $T_{\lambda,2}$, i.e.

$$P_1(\alpha_1, \alpha_2) = \alpha_1, \qquad P_2(\alpha_1, \alpha_2) = \alpha_2. \tag{5.4}$$

The module algebra structure included in $U_\lambda(\mathfrak{iso}_\omega(2)) = U(\mathfrak{so}_\omega(2)) \triangleright\!\!\blacktriangleleft F(T_{\lambda,2})$ determines the action of $SO_\omega(2)$ over $T_{\lambda,2}$ by means of the vector field

$$\hat{J}_{\omega,\lambda} = -\left[\frac{1 - \mathrm{e}^{-2\lambda P_2}}{2\lambda} + \frac{1}{2}\lambda\omega P_1^2\right]\frac{\partial}{\partial P_1} + \omega P_1 \frac{\partial}{\partial P_2}. \tag{5.5}$$

Prior to determining the flow associated with $\hat{J}_{\omega,\lambda}$ we can obtain a first qualitative information by identifying the fixed points of $\hat{J}_{\omega,\lambda}$. When $\omega \neq 0$ only the origin $(0, 0)$ is an equilibrium point, but if $\omega = 0$ the set of fixed points is the straight line of equation $\alpha_2 = 0$. The deformation due to λ does not change the character of these points. Thus, the point is elliptic if $\omega > 0$ and hyperbolic if $\omega < 0$.

The one-form

$$\eta_\rho = \rho\left[\omega P_1 \,\mathrm{d}P_1 + \left(\frac{1 - \mathrm{e}^{-2\lambda P_2}}{2\lambda} + \frac{1}{2}\lambda\omega P_1^2\right)\mathrm{d}P_2\right], \qquad \rho \in F(T_{\lambda,2}), \tag{5.6}$$

verifies $\hat{J}_{\omega,\lambda} \rfloor \eta_\rho = 0$. Taking $\rho_0 = \lambda^2 \mathrm{e}^{\lambda P_2}$ one obtains that η_{ρ_0} is exact and invariant under $\hat{J}_{\omega,\lambda}$. Adding a constant and rescaling the invariant function under the action of J

$$h = \frac{1}{2}\omega\lambda^2 P_1^2 \mathrm{e}^{\lambda P_2} + \cosh(\lambda P_2), \tag{5.7}$$

we obtain a central element of $U_\lambda(\mathfrak{iso}_\omega(2))$

$$\mathrm{C}_{\omega,\lambda} = \omega P_1^2 \mathrm{e}^{\lambda P_2} + 2\frac{\cosh(\lambda P_2) - 1}{\lambda^2} = \omega P_1^2 \mathrm{e}^{\lambda P_2} + \frac{4}{\lambda^2}\sinh^2\left(\frac{\lambda}{2}P_2\right) \tag{5.8}$$

such that in the limit $\lambda \to 0$ we recover the nondeformed Casimir $\mathrm{C}_{\omega,0} = \omega P_1^2 + P_2^2$ of $\mathfrak{iso}_\omega(2)$.

The computation of the trajectories of $\hat{J}_{\omega,\lambda}$ requires one to solve the equation system

$$\begin{aligned} \dot{\alpha_1} &= -\left[\frac{1-\mathrm{e}^{-2\lambda\alpha_2}}{2\lambda} + \frac{1}{2}\lambda\omega\alpha_1^2\right], \\ \dot{\alpha_2} &= \omega\alpha_1. \end{aligned} \tag{5.9}$$

The integral curve γ can be expressed as

$$\begin{aligned} \alpha_1(t) &= -\frac{1}{\lambda}\frac{\sinh(\lambda\beta)\mathrm{S}_\omega(t)}{\cosh(\lambda\beta)+\sinh(\lambda\beta)\mathrm{C}_\omega(t)}, \\ \alpha_2(t) &= \frac{1}{\lambda}\ln[\cosh(\lambda\beta)+\sinh(\lambda\beta)\mathrm{C}_\omega(t)]. \end{aligned} \tag{5.10}$$

Using the fact that $\Phi^t_{\omega,\lambda}(\gamma(\tau)) = \gamma(\tau+t)$ the flow can be evaluated

$$\begin{aligned} \Phi^t_{\omega,\lambda}(\alpha_1,\alpha_2) = \Big(& \{2\alpha_1\lambda\mathrm{e}^{\lambda\alpha_2}\mathrm{C}_\omega(t) + (\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2} - \sinh(\lambda\alpha_2))\mathrm{S}_\omega(t)\} \\ & \times\{2\lambda\cosh(\lambda\alpha_2) + \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2} + (2\lambda\sinh(\lambda\alpha_2) - \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(t) \\ & + 2\omega\lambda^2\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(t)\}^{-1}, \frac{1}{\lambda}\ln\Big[\cosh(\lambda\alpha_2) + \frac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2} \\ & + (\sinh(\lambda\alpha_2) - \tfrac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(t) + \omega\lambda\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(t)\Big]\Big). \end{aligned} \tag{5.11}$$

In the limit $\lambda \to 0$ we recover the linear flow

$$\Phi^t_{\omega,0}(\alpha_1,\alpha_2) = (\mathrm{C}_\omega(t)\alpha_1 - \mathrm{S}_\omega(t)\alpha_2, \omega\mathrm{S}_\omega(t)\alpha_1 + \mathrm{C}_\omega(t)\alpha_2), \tag{5.12}$$

which corresponds to the nondeformed action given by the vector field $\hat{J}_{\omega,0} = -P_2\partial_{P_1} + \omega P_1\partial_{P_2}$.

Note that although expression (5.11) has been obtained for $\omega > 0$ and $\lambda > 0$ it can be proved that it is also valid for the remaining values. In some sense, the flow has an 'analytic' dependence on the parameters ω and λ, which allows one to extend the results obtained in a region of the parameters space to the whole of it.

The flow (5.11) is globally defined when $\omega \geqslant 0$ but is only local for $\omega < 0$ as it is easy to prove considering the integral curve passing through a point such as $(0, \alpha_2)$: since the logarithm argument has to be positive one obtains the inequality

$$\cosh(\lambda\alpha_2) + \sinh(\lambda\alpha_2)\mathrm{C}_\omega(t) > 0, \tag{5.13}$$

if the product $\lambda\alpha$ is negative the definition interval of t is bounded

$$t \in (-\mathrm{C}_\omega^{-1}(-\coth(\lambda\alpha_2)), \mathrm{C}_\omega^{-1}(-\coth(\lambda\alpha_2))). \tag{5.14}$$

The flow (5.11) describes the action of $SO_\omega(2)$ over $T_{\lambda,2}$

$$\mathrm{e}^{tJ} \triangleright (\alpha_1,\alpha_2) = \Phi^t_{\lambda,\omega}(\alpha_1,\alpha_2), \tag{5.15}$$

which decomposes the space $T_{\lambda,2}$ on the strata of orbits depending on the values of ω:

- *Case* $\omega < 0$
  - ∗ A stratum with only one orbit with only one point: $(0, 0)$. The isotropy group is $SO_\omega(2)$.
  - ∗ Four orbits determined by the points

$$\begin{aligned} &\left(\frac{1-\mathrm{e}^{-\lambda}}{\lambda\sqrt{-\omega}}, 1\right), \qquad \left(\frac{1-\mathrm{e}^{\lambda}}{\lambda\sqrt{-\omega}}, -1\right), \qquad \left(-\frac{1-\mathrm{e}^{-\lambda}}{\lambda\sqrt{-\omega}}, 1\right), \\ &\left(-\frac{1-\mathrm{e}^{\lambda}}{\lambda\sqrt{-\omega}}, -1\right). \end{aligned} \tag{5.16}$$

    These orbits are isomorphic to $\mathbb{R}$ and have the point $(0, 0)$ as an accumulation point.
  - ∗ The remaining points of $T_{\lambda,2}$ constitute another stratum fibred by orbits diffeomorphic to $\mathbb{R}$ since they are branches of deformed hyperbolae.

- *Case* $\omega = 0$
  - $*$ A stratum is constituted by the points $(\alpha_1, 0)$, each of them is an orbit.
  - $*$ The orbits $\{(\alpha_1, \alpha_2) | \alpha_1 \in \mathbb{R}, \alpha_2 \neq 0$ but fixed$\}$ determine a stratum.
- *Case* $\omega > 0$
  - $*$ The point $(0, 0)$ constitutes the only orbit of this stratum.
  - $*$ The orbits diffeomorphic to the circle determine another stratum.

Note that the deformation associated with λ does not give qualitative changes with respect to the nondeformed case. Summarizing, we can say that the quotient spaces are isomorphic:

$$T_{\lambda,2}/SO_\omega(2) \simeq T_{0,2}/SO_\omega(2). \tag{5.17}$$

5.2. Regular co-spaces

Once the flow associated with the bicrossproduct structure of $U_\lambda(\mathfrak{iso}_\omega(2)))$ is known, we can study the regular co-spaces $(F_\lambda(ISO_\omega(2)), \prec, U_\lambda(\mathfrak{iso}_\omega(2)))$ and $(F_\lambda(ISO_\omega(2)), \succ, U_\lambda(\mathfrak{iso}_\omega(2)))$, which are basic elements characterizing the induced representations of $U_\lambda(\mathfrak{iso}_\omega(2)))$.

The structures of both co-spaces are easily deduced combining theorem 3.3 with expression (5.11) of the flow of $\hat{J}$. Firstly, remember that $(F_\lambda(ISO_\omega(2))$ can be described by elements of the form

$$\phi(\alpha_1, \alpha_2), \qquad \phi \in F(SO_\omega(2)), \qquad (\alpha_1, \alpha_2) \in T_{\lambda,2}, \tag{5.18}$$

instead of monomials $\varphi^q a_1^r a_2^s$.

(1) For the right coregular module $(F_\lambda(ISO_\omega(2)), \prec, U_\lambda(\mathfrak{iso}_\omega(2)))$ we have that

$$\begin{aligned}
&[\phi(\alpha_1, \alpha_2)] \prec \mathrm{e}^{tJ} = \phi(\mathrm{e}^{tJ}\cdot)(\alpha_1, \alpha_2),\\
&[\phi(\alpha_1, \alpha_2)] \prec P_1 = \phi\{2\alpha_1\lambda \mathrm{e}^{\lambda\alpha_2}\mathrm{C}_\omega(\varphi) + (\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2}\\
&\qquad - 2\sinh(\lambda\alpha_2))\mathrm{S}_\omega(\varphi)\}\{2\lambda\cosh(\lambda\alpha_2) + \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2}) + (2\lambda\sinh(\lambda\alpha_2)\\
&\qquad - \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(\varphi) + 2\omega\lambda^2\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(\varphi)\}^{-1}(\alpha_1, \alpha_2),\\
&[\phi(\alpha_1, \alpha_2)] \prec P_2 = \phi\frac{1}{\lambda}\ln\left[\cosh(\lambda\alpha_2) + \frac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2} + \left(\sinh(\lambda\alpha_2) - \frac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2}\right)\mathrm{C}_\omega(\varphi)\right.\\
&\qquad \left. + \omega\lambda\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(\varphi)\right](\alpha_1, \alpha_2).
\end{aligned}$$

The dot stands for the argument of the function $\phi = \phi(\cdot)$. Developing in power series of t and considering the first order in the first expression and the multiplication and derivation operators associated with the basis $\varphi^q a_1^r a_2^s$ we can write the action of the generators of $U_\lambda(\mathfrak{iso}_\omega(2))$ over an arbitrary 'function' $f \in F_\lambda(ISO_\omega(2))$ only making the changes

$$\varphi \to \bar{\varphi}, \qquad a_i \to \frac{\partial}{\partial a_i} \equiv \partial_{a_i}. \tag{5.19}$$

In this way we obtain

$$\begin{aligned}
&f \prec J = \partial_\varphi f,\\
&f \prec P_1 = \left\{2\lambda\partial_{a_1}\mathrm{e}^{\lambda\partial_{a_2}}\mathrm{C}_\omega(\bar{\varphi}) + \left(\omega\lambda^2\frac{\partial^2}{\partial a_1^2}\mathrm{e}^{\lambda\partial_{a_2}} - 2\sinh(\lambda\partial_{a_2})\right)\mathrm{S}_\omega(\bar{\varphi})\right\}\\
&\qquad \times\left\{2\lambda\cosh(\lambda\partial_{a_2}) + \omega\lambda^3\frac{\partial^2}{\partial a_1^2}\mathrm{e}^{\lambda\partial_{a_2}} + \left(2\lambda\sinh(\lambda\partial_{a_2}) - \omega\lambda^3\frac{\partial^2}{\partial a_1^2}\mathrm{e}^{\lambda\partial_{a_2}}\right)\mathrm{C}_\omega(\bar{\varphi})\right.\\
&\qquad \left. + 2\omega\lambda^2\partial_{a_1}\mathrm{e}^{\lambda\partial_{a_2}}\mathrm{S}_\omega(\bar{\varphi})\right\}^{-1} f,
\end{aligned}$$

$$f \prec P_2 = \frac{1}{\lambda}\ln\bigg[\cosh(\lambda\partial_{a_2}) + \frac{1}{2}\omega\lambda^2\frac{\partial^2}{\partial a_1^2}e^{\lambda\partial_{a_2}} + \bigg(\sinh(\lambda\partial_{a_2}) - \frac{1}{2}\omega\lambda^2\frac{\partial^2}{\partial a_1^2}e^{\lambda\partial_{a_2}}\bigg)C_\omega(\bar\varphi)$$
$$+ \omega\lambda\partial_{a_1}e^{\lambda\partial_{a_2}}S_\omega(\bar\varphi)\bigg]f.$$

From the above result the action of the generator J over the ordering monomials $\varphi^q a_1^r a_2^s$ is easily evaluated:

$$(\varphi^q a_1^r a_2^s) \prec J = q\varphi^{q-1}a_1^r a_2^s, \tag{5.20}$$

but this is not the case with P_1 and P_2, except in the nondeformed case where the action reduces to

$$f \prec P_1 = \bigg(C_\omega(\bar\varphi)\frac{\partial}{\partial a_1} - S_\omega(\bar\varphi)\frac{\partial}{\partial a_2}\bigg)f, \qquad f \prec P_2 = \bigg(\omega S_\omega(\bar\varphi)\frac{\partial}{\partial a_1} + C_\omega(\bar\varphi)\frac{\partial}{\partial a_2}\bigg)f. \tag{5.21}$$

So, when $\lambda = 0$ one obtains

$$\begin{aligned}(\varphi^q a_1^r a_2^s) \prec P_1 &= r\varphi^q C_\omega(\varphi)a_1^{r-1}a_2^s - s\varphi^q S_\omega(\varphi)a_1^r a_2^{s-1},\\ (\varphi^q a_1^r a_2^s) \prec P_1 &= \omega r\varphi^q S_\omega(\varphi)a_1^{r-1}a_2^s + s\varphi^q C_\omega(\varphi)a_1^r a_2^{s-1}.\end{aligned} \tag{5.22}$$

(2) The description of the left coregular module $(F_\lambda(ISO_\omega(2)), \succ, U_\lambda(\mathfrak{iso}_\omega(2)))$ is performed in an analogous way. Firstly, considering theorem 3.3 one obtains

$$\begin{aligned}e^{tJ} \succ [\phi(\alpha_1,\alpha_2)] &= \phi(\cdot e^{tJ})\Big(\{2\lambda\alpha_1 e^{\lambda\alpha_2}C_\omega(t) + (\omega\lambda^2\alpha_1^2 e^{\lambda\alpha_2} - 2\sinh(\lambda\alpha_2))S_\omega(t)\}\\ &\quad\times\{2\lambda\cosh(\lambda\alpha_2) + \omega\lambda^3\alpha_1^2 e^{\lambda\alpha_2} + (2\lambda\sinh(\lambda\alpha_2) - \omega\lambda^3\alpha_1^2 e^{\lambda\alpha_2})C_\omega(t)\\ &\quad + 2\omega\lambda^2\alpha_1 e^{\lambda\alpha_2}S_\omega(t)\}^{-1}, \frac{1}{\lambda}\ln\Big[\cosh(\lambda\alpha_2) + \frac{1}{2}\omega\lambda^2\alpha_1^2 e^{\lambda\alpha_2}\\ &\quad + \big(\sinh(\lambda\alpha_2) - \tfrac{1}{2}\omega\lambda^2\alpha_1^2 e^{\lambda\alpha_2}\big)C_\omega(t) + \omega\lambda\alpha_1 e^{\lambda\alpha_2}S_\omega(t)\Big]\Big),\\ P_1 \succ [\phi(\alpha_1,\alpha_2)] &= \alpha_1\phi(\alpha_1,\alpha_2),\\ P_2 \succ [\phi(\alpha_1,\alpha_2)] &= \alpha_2\phi(\alpha_1,\alpha_2).\end{aligned}$$

With the same arguments in the case of the right regular co-space the action of the generators can be written in terms of the multiplication and derivation operators

$$\begin{aligned}J \succ f &= \bigg[\frac{\partial}{\partial\varphi} - \bar a_1\frac{1 - e^{-2\lambda\frac{\partial}{\partial a_2}}}{2\lambda} + \omega\bar a_2\frac{\partial}{\partial a_1} - \frac{\lambda}{2}\omega\bar a_1\frac{\partial^2}{\partial a_1^2}\bigg]f,\\ P_i \succ f &= \frac{\partial}{\partial a_i}f, \qquad i = 1, 2,\end{aligned} \tag{5.23}$$

which allows one to obtain the action of the generators over the basis $\varphi^q a_1^r a_2^s$

$$\begin{aligned}J \succ (\varphi^q a_1^r a_2^s) &= q\varphi^q a_1^r a_2^{s-1} - \frac{\lambda}{2}\omega r(r-1)\varphi^q a_1^{r-1}a_2^s + \omega r\varphi^q a_1^{r-1}a_2^{s+1}\\ &\quad - \frac{1}{2\lambda}\varphi^q a_1^{r+1}a_2^s + \frac{1}{2\lambda}\varphi^q a_1^{r+1}(a_2+2\lambda)^s s\varphi^q a_1^r a_2^{s-1},\\ P_1 \succ (\varphi^q a_1^r a_2^s) &= r\varphi^q a_1^{r-1}a_2^s,\\ P_2 \succ (\varphi^q a_1^r a_2^s) &= s\varphi^q a_1^r a_2^{s-1}.\end{aligned} \tag{5.24}$$

The subalgebra $\mathcal{A}_{\omega,\lambda} = \langle a_1, a_2\rangle$ of $F_\lambda(ISO_\omega(2))$, which is not a Hopf subalgebra, is stable under the previous action. The explicit action of the generators of $U_\lambda(\mathfrak{iso}_\omega(2))$ over a generic element $\psi(a_1, a_2)$ of $\mathcal{A}_{\omega,\lambda}$ in terms of operators adapted to the basis $a_1^r a_2^s$ is

$$\begin{aligned} J \triangleright \psi(a_1, a_2) &= \left(-\bar{a}_1 \frac{1 - \mathrm{e}^{-2\lambda \frac{\partial}{\partial a_2}}}{2\lambda} + \omega\bar{a}_2 \frac{\partial}{\partial a_1} - \frac{\lambda}{2}\omega\bar{a}_1 \frac{\partial^2}{\partial a_1^2}\right)\psi(a_1, a_2), \\ P_i \triangleright \psi(a_1, a_2) &= \frac{\partial}{\partial a_i}\psi(a_1, a_2), \qquad i = 1, 2. \end{aligned} \tag{5.25}$$

The generators a_1 and a_2 of the co-space $(\mathcal{A}_{\omega,\lambda}, \triangleright, U_\lambda(\mathfrak{iso}_\omega(2)))$ verify the commutation relation

$$[a_1, a_2] = \lambda a_1 \tag{5.26}$$

defining a noncommutative geometry except in the nondeformed limit where the corresponding classical geometries are recovered.

The action of the Casimir (5.8) of $U_\lambda(\mathfrak{iso}_\omega(2))$ on $(\mathcal{A}_{\omega,\lambda}, \triangleright, U_\lambda(\mathfrak{iso}_\omega(2)))$ is given by

$$\mathrm{C}_{\omega,\lambda} \triangleright f(a_1, a_2) = \left[\omega \frac{\partial^2}{\partial a_1^2}\mathrm{e}^{\lambda\frac{\partial}{\partial a_2}} + \frac{4}{\lambda^2}\sinh^2\left(\frac{\lambda}{2}\frac{\partial}{\partial a_2}\right)\right] f(a_1, a_2), \tag{5.27}$$

or explicitly

$$\mathrm{C}_{\omega,\lambda} \triangleright f(a_1, a_2) = \omega\frac{\partial^2}{\partial a_1^2} f(a_1, a_2 + \lambda) + \frac{1}{\lambda^2}[f(a_1, a_2 + \lambda) + f(a_1, a_2 - \lambda) - 2f(a_1, a_2)]. \tag{5.28}$$

This last expression shows the effect of the deformation transforming one of the derivatives on a finite difference operator.

An interesting problem is the solution of the wave equations associated with this two-parameter family of Casimir operators (in this context see, for instance [35] and references therein). Considering the group $T_{\lambda,2}$ included inside $\mathcal{A}_{\omega,\lambda}$ by means of the exponential map, the action of the Casimir over the elements $(\alpha_1, \alpha_2) \in T_{\lambda,2}$ is given by

$$\mathrm{C}_{\omega,\lambda} \triangleright (\alpha_1, \alpha_2) = \left[\omega\alpha_1^2\mathrm{e}^{\lambda\alpha_2} + \frac{4}{\lambda^2}\sinh^2\left(\frac{\lambda}{2}\alpha_2\right)\right](\alpha_1, \alpha_2), \tag{5.29}$$

which suggests interpreting (α_1, α_2) as a 'plane wave'.

5.3. Representations of $U_\lambda(\mathfrak{iso}_\omega(2))$

The representation induced by the character of $\mathcal{L}$

$$1 \dashv (P_1^{n_1} P_2^{n_2}) = \alpha_1^{n_1}\alpha_2^{n_2} \tag{5.30}$$

(or, in other words, the representation induced by $(\alpha_1, \alpha_2) \in T_{\lambda,2}$) is obtained as follows. Theorem 3.4 gives the following result for the action of the generators in the induced representation

$$\begin{aligned} \phi \dashv J &= \phi', \\ \phi \dashv P_1 &= \phi\{2\lambda\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{C}_\omega(\varphi) + (\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2} - 2\sinh(\lambda\alpha_2))\mathrm{S}_\omega(\varphi)\}\{2\lambda\cosh(\lambda\alpha_2) \\ &\qquad + 2\omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2} + (2\lambda\sinh(\lambda\alpha_2) - \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(\varphi) + 2\omega\lambda^2\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(\varphi)\}^{-1}, \\ \phi \dashv P_2 &= \phi\frac{1}{\lambda}\ln\left[\cosh(\lambda\alpha_2) + \frac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2}\right. \\ &\qquad \left. + (\sinh(\lambda\alpha_2) - \tfrac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(\varphi) + \omega\lambda\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(\varphi)\right]. \end{aligned}$$

In the limit $\lambda \to 0$ we recover the more familiar expressions

$$\begin{aligned}
&\phi \dashv J = \phi',\\
&\phi \dashv P_1 = \phi(\alpha_1 \mathrm{C}_\omega(\varphi) - \alpha_2 \mathrm{S}_\omega(\varphi)),\\
&\phi \dashv P_2 = \phi(\omega\alpha_1 \mathrm{S}_\omega(\varphi) + \alpha_2 \mathrm{C}_\omega(\varphi)).
\end{aligned} \tag{5.31}$$

The local representations obtained inducing with the character of $U(\mathfrak{so}_\omega(2))$ given by

$$J^m \vdash 1 = c^m, \tag{5.32}$$

are

$$\begin{aligned}
\mathrm{e}^{tJ} \vdash (\alpha_1, \alpha_2) = \mathrm{e}^{tc}\Bigg(&\{2\lambda\alpha_1 \mathrm{e}^{\lambda\alpha_2}\mathrm{C}_\omega(t) + (\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2} - 2\sinh(\lambda\alpha_2))\mathrm{S}_\omega(t)\}\\
&\times \{2\lambda\cosh(\lambda\alpha_2) + \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2} + (2\lambda\sinh(\lambda\alpha_2) - \omega\lambda^3\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(t)\\
&+ 2\omega\lambda^2\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(t)\}^{-1}, \frac{1}{\lambda}\ln\bigg[\cosh(\lambda\alpha_2) + \frac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2}\\
&+ (\sinh(\lambda\alpha_2) - \tfrac{1}{2}\omega\lambda^2\alpha_1^2\mathrm{e}^{\lambda\alpha_2})\mathrm{C}_\omega(t) + \omega\lambda\alpha_1\mathrm{e}^{\lambda\alpha_2}\mathrm{S}_\omega(t)\bigg]\Bigg),\\
P_i \vdash (\alpha_1, \alpha_2) = \alpha_i&(\alpha_1, \alpha_2), \qquad i = 1, 2,
\end{aligned}$$

which can be written in an equivalent way using an arbitrary element, $\psi(a_1, a_2)$, of $U(\mathfrak{t}_{\lambda,2})$, as

$$\begin{aligned}
&J \vdash \psi(a_1, a_2) = \left[c - \bar{a}_1 \frac{1 - \mathrm{e}^{-2\lambda\frac{\partial}{\partial a_2}}}{2\lambda} + \omega\bar{a}_2 \frac{\partial}{\partial a_1} - \frac{\lambda}{2}\omega\bar{a}_1\frac{\partial^2}{\partial a_1^2}\right]\psi(a_1, a_2),\\
&P_i \vdash \psi(a_1, a_2) = \frac{\partial}{\partial a_i}\psi(a_1, a_2), \qquad i = 1, 2.
\end{aligned} \tag{5.33}$$

Notice that if we take $c = 0$, which is equivalent to considering the character determined by the counit of $U(\mathfrak{so}_\omega(2))$, the action (5.25) of the co-space $(\mathcal{A}_{\omega,\lambda}, \triangleright, U_\lambda(\mathfrak{iso}_\omega(2)))$ is recovered.

The Casimir action for the local representation coincides with those given by (5.27) since only the generators P_i are presented.

It is also worth noting that in this case in the limit $\lambda \to 0$ we obtain

$$\begin{aligned}
&J \vdash \psi(a_1, a_2) = \left[c - a_1\frac{\partial}{\partial a_2} + \omega a_2 \frac{\partial}{\partial a_1}\right]\psi(a_1, a_2),\\
&P_i \vdash \psi(a_1, a_2) = \frac{\partial}{\partial a_i}\psi(a_1, a_2), \qquad i = 1, 2,
\end{aligned} \tag{5.34}$$

which it is in agreement with the results for local representations [36].

6. Concluding remarks

The induction procedure that we have used here is not, strictly speaking, a generalization of Mackey's induction method for Lie groups. The concept of co-space generalizes in an algebraical way the concept of G-space (G being a transformation group) and is connected with the induced representations in some sense similar to the nondeformed case. In [6] we presented a more general method that, obviously, can also be used for bicrossproduct algebras. In [6] the knowledge of pairs on the dual basis of the corresponding Hopf algebra and its dual is required, however, in the method used in this work this requirement is not necessary.

We have made use of the vector fields to compute commutators. For bicrossproduct Hopf algebras, as used in this work $H = \mathcal{K} \bowtie \mathcal{L}$, there is a connection between the representations of H induced by the characters of L and one-parameter flows. This relation allows one to associate with quantum bicrossproduct groups dynamical systems. Here we have only sketched the situation that is to be analysed in greater detail in [37].

The equations associated with the Casimir operators, as in the nondeformed case, will give the behaviour of the 'deformed' quantum systems. A procedure for this solution can be found in [35]. Note that q-special polynomials and q-functions may appear as solutions of these q-Casimir equations.

From the point of view of applications the physics of unitarity representations is an important question. To discuss this problem it is necessary to fix a $*$-structure on the Hopf algebra and a scalar product on the vector space that supports the representation. Following the method described in section 2 we can select the standard $*$-structures on the factors of $H = U(\mathfrak{k}) \bowtie F(L)$ to get a $*$-structure on the algebra sector of H. Then, the first expression in (3.27) says that the induced action of group-like elements $k \in U(\mathfrak{k}) \subset H$ is nothing other than the (right-) regular one and henceforth, to ensure unitarity, we have to choose the Hilbert space structure on $F(K)$ corresponding to the Haar measure on K. On the other hand, from second formula in (3.27) we see that elements $\lambda \in F(K) \subset H$ act by means of a multiplicative factor. Therefore, to be consistent with the $*$-structure, the deformation parameter must be real but this choice leads to an unsolved problem: some divergent factors arise due to the local character of the action of K on L which only turns out to be global for vanishing deformation parameter values.

The equivalence of the induced representations is given by theorem 3.4, which establishes a correspondence between classes of induced representations and orbits of L under the action of K. This result is analogous to the Kirillov orbits method [38].

The problem of the irreducibility of the representations is still open. Partial results for particular cases have been obtained (see [4] for the (1 + 1) κ-Galilei algebra and [33] for the quantum extended (1 + 1) Galilei algebra).

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